Available online at www.sciencedirect.com

ScienceDirect

Physics Procedia 15 (2011) 102–105

Physics Procedia

Studies of adaptive networks with preferred degree

R.K.P. Zia, Wenjia Liu, Shivakumar Jolad, and B. Schmittmann

Physics Department, Virginia Tech, Blacksburg, VA, 24061-0435, USA

Abstract

We study a simple model involving adaptive networks in which the nodes add or cut links to other nodes according to a set preferred degree, κ. This behavior seems more natural for human beings as they form a circle of a preferred number of friends or contacts. In the simplest model, a node with degree k will add (cut) a link with probability $w_+(k)$ $(1 - w_+(k))$. Several forms of w_+ are considered, e.g., a step function that drops abruptly from unity to zero as k increases beyond κ. Using simulations, we find the degree distribution in the steady state. Unexpectedly, it is not a Gaussian (around κ). We are able to find an approximate theory which explains these distributions quite well. Introducing a second network and coupling the two in various ways, we find both understandable and puzzling features. In the third part, we consider overlaying an SIS model of epidemics on a single adaptive network, allowing κ to depend on the fraction of the infected population. The gross features of the resulting steady states can be well explained by a mean field like theory, balancing the rates of recovery and infection. Various avenues for further investigations are proposed.

Keywords: Adaptive network, Co-evolving network, Social network, Epidemics

Introduction. From the internet to neuronal architectures and from facebook to power grids, many biological, physical, and social systems can be characterized in the language of networks [1]. Their emergence has been truly transformative for modern societies. Yet, many of the underlying fundamental aspects of networks remain poorly understood. Does function really follow from topology in, e.g., biological networks? How can we use network science to manage critical infrastructures, such as airline networks or power grids, better? Can we data-mine the network structure underlying wikipedia and identify the next emerging, major ideas?

Naturally, time-dependent phenomena (or 'emergent behaviors') play a huge role in network science [2]. After the initial work on characterizing static networks, the next step consisted in assigning variables to the nodes or links and modeling dynamic processes on static networks. We refer to this type of study as 'dynamics on networks.' A simple example is traffic on a network of roads, in the absence of road construction or closures. If the network itself becomes dynamic, by creating or deleting links or nodes, one investigates the 'dynamics of networks.' In our example, roads may be added or closed but without any attention to usage. The next level of complexity involves adaptive networks: the dynamics of the variables on the nodes or links is coupled to the dynamics of the network itself, so that a nontrivial feedback loop emerges. In the traffic example, usage is now monitored and new lanes or roads are opened if there is enough demand for them, and unused roads are left to decay. And finally, one can ask how dynamic processes on adaptive networks play out if they involve interactions between two or more networks. Motorists are now following traffic predictions on their smart phones, interacting with wireless networks.

The detailed analysis of such interconnectivity in specific networks is the realm of engineering. As physicists, we are more focused on understanding underlying universal features, extracted from the study of simple models. In the following, we focus on a very simple model involving adaptive networks, cast in the language of epidemics in a

doi:10.1016/j.phpro.2011.06.008

society where individuals have a preferred degree, κ (i.e., a preference for a certain number of 'friends'). Our study begins with modeling a single, homogeneous network with just one κ. Here, the nodes play only the role of adding or cutting links, according to specified rates controlled by κ. When such a dynamic network reaches a steady state, remarkably, the degree distribution is not, in general, a Gaussian (around κ). A simple argument leads to surprisingly good agreement with simulation results. Introducing a second network (i.e., another group of nodes) and coupling the two in various ways, we investigate the degree distributions for both intra- and inter-group links, with understandable and puzzling results. Finally, we return to a single network but allow the nodes to be endowed with degrees of freedom, by considering a simple contact process: the SIS model of epidemics[3]. In contrast to the numerous studies of such processes on regular lattices or scale-free networks, our work here is more realistic in two aspects: First, we model the epidemic on a network which more closely resembles a set of social contacts. Second, our network is adaptive, i.e., it allows us to model a natural human response to the onset of an epidemic, namely, cutting down on the number of contacts. Interesting phenomena emerge as we let κ depend on, say, the fraction of infected individuals in the system. In this brief article, we report preliminary results and outline future investigations of other aspects.

Modeling dynamic networks with degree preference. Focusing first on a single network, we consider N nodes (individuals), all behaving in the same manner, and study only the dynamics of the links. In one attempt, we choose a random node and find its degree, k (the number of links it has). Then, with probability $w_+(k)$, we create a new link, to a randomly chosen node not already linked to it. Similarly, we cut a randomly chosen, existing link with probability $w_-(k)$. Otherwise (probability $1 - w_+ - w_-$), we keep the status quo. At the simplest level, we choose $w_- = 1 - w_+$. By having w_+ decrease (say, monotonically) from unity to zero with increasing k, we introduce the preferred degree, κ, as a parameter: $w_+(k|\kappa)$. We can further model the 'flexibility' of an individual through how abruptly w_+ drops as k passes κ. The simplest choice, corresponding to the most inflexible behavior, is a step function: $w_+ = 1$ for $k \leq \kappa$ and 0 otherwise. Such individuals will make new links as long as they have κ or fewer 'friends' and attempt to cut one, as soon as they have more than κ links. We have also simulated models with more flexible w_+'s, e.g., a Fermi-Dirac function, $\left[1 + e^{-\beta\kappa}\right] / \left[1 + e^{\beta(k-\kappa)}\right]$, in which β is obviously a measure of 'inflexibility.' Turning to the modeling of two (or more) groups, the possibilities seem boundless. The most obvious and simple generalizations are different sizes ($N_1 \neq N_2$) and preferences ($\kappa_1 \neq \kappa_2$). For example, our society consists of ~25% introverts, who prefer fewer contacts than the extroverts. Even within these simple generalizations, an additional rule is needed to model typical behavior: When adding or cutting a link is desired, an individual may act differently depending on whether the partner is from the same or from a different group. Many examples of this differentiation exist, leading to say, the self-insulating discourse of political groups in the present US society. Here, we again focus on the simplest possibility: a one parameter model. When the node acts on a link, it chooses an *inter*-group partner with probability χ. Clearly, this parameter controls the level of cross-links between the two groups, and effectively measures the level of interaction between them. Of course, members of the two groups may have different behaviors, so that $\chi_1 \neq \chi_2$ in general. In our simulations, a Monte Carlo step (MCS) is defined as N (or $N_1 + N_2$) such attempts, so that, on the average, each node has been chosen once during an MCS.

Epidemics on an adaptive network. Finally, let us introduce node-variables, S/I (for Susceptible/Infected), but focus only on a single, adaptive network to begin with. These variables are updated in a standard fashion. An I node is 'flipped' to S with probability μ, representing the spontaneous recovery from being infected (say, with a cold). For an S node, we assign a probability λ for it to be infected by *any one* of its infected contacts. Thus, if it is linked to k_I infected nodes, then it will stay as S with probability $(1 - \lambda)^{k_I}$ (and flip otherwise). Clearly, the system has an absorbing state, in which all are healthy. On the other hand, for large enough λ, a finite fraction, f, of the population can remain infected for an extended period of time – the active state. In standard SIS models, this threshold for an epidemic (denoted by λ_c) is sharp in the $N \to \infty$ limit, and depends on μ as well as, e.g., the dimensionality of the system[4]. Turning to our adaptive network, we note that the typical time scales for adding/cutting contacts are larger than those associated with a disease. So, we generally perform $n_{SI} \gtrsim 10$ MCS updating the nodes between sucessive MCS of link updates. In this language, $n_{SI} = \infty$ corresponds to the standard SIS model on a static network. While using such a system to model a forest (where a healthy tree is unable to move away from a neighboring infected tree) may be appropriate, it is arguable less applicable for humans with social contacts. Naturally, a healthy individual is likely to avoid contacts with those infected. In the case of a visible disease (e.g., cold), such actions may be taken on a case by case basis. To investigate the effects of infection-avoidance on an epidemic, several authors [5] 'rewired' their network, by moving a link with an I node to a link with an S node. Instead of imposing such a constraint, we model human response by letting κ depend on the epidemic. Again, there are many possible ways to introduce such a dependence. To model a disease that is not overtly obvious (e.g., AIDS), we let an individual's preference for contacts decrease when he/she discovers how widespread the epidemic has become. In other words, we allow κ to depend only on f. This is clearly a response at a 'global' level, as opposed to a more 'local' response in which an individuals adjusts his/her κ based on the condition of his/her circle of friends alone. Again, many possible choices of $\kappa(f)$ present themselves. So far, we considered three types of 'fear factors': (a) the 'fearless,' where κ remains at $\kappa_{\max}$ until the epidemic is widespread (say, 40% infected) and then drops abruptly to $\kappa_{\min}$ (representing only family members and caretakers); (b) the 'moderate,' where κ drops linearly from $\kappa_{\max}$ to $\kappa_{\min}$; and (c) the 'nosophobic' (or 'agoraphobic') with $\kappa = \kappa_{\max} e^{-\alpha f}$. As shown below, gross features such as thresholds and $f(\lambda)$ can be understood in terms of a simple mean field theory, but explaining details such as large fluctuations near λ_c and degree distributions remains challenging. Beyond this simple model, we envisage studying two or more groups (e.g., extroverts and introverts), in order to gain some insight into epidemic spreading/control in adaptive networks.

Simulation and analytic results. We perform simple Monte Carlo simulations on the three systems (one/two dynamic networks with static nodes and S/I on a single adaptive network) described above and measure a number of quantities. Most unexpected in the single network case is the degree distribution in the steady state, $\rho(k)$. With $N = 1000$ and $\kappa = 250$, we find that, instead of being a Gaussian around κ, it depends on the details of w_+. In the 'inflexible' case, $\rho(k)$ is a pure *double exponential*, peaking at κ. For $\beta < \infty$, it is Gaussian around κ within a width of $1/\beta$, crossing over to two (symmetric) exponential tails. This behavior can be understood from a simple argument (which resembles detailed balance): The rate a node with degree k loses a link is $1 - w_+(k)$ *plus* 1/2 (from other nodes cutting their links). Similarly, it can gain with rate $w_+ + 1/2$, so that

$$\rho(k+1)/\rho(k) = [w_+(k) + 1/2] / [w_-(k+1) + 1/2] \tag{1}$$

In particular, with inflexible individuals this ratio is just 3 for $k < \kappa$ and 1/3 for $k > \kappa$. For all cases which we have examined (various N's, κ's, and w's), this prediction is in excellent agreement with the simulations. Extending this study to two communities, we considered a range of possible N, κ, χ's. A similar argument for a system with two groups (having the same N's) leads to the modification $1/2 \rightarrow (1 - \chi_1 + \chi_2)/2$ for $\rho_1(k)$, etc. and also provides excellent predictions. The main puzzles concern the separate degree distributions for intra-group links (ρ_{11}, ρ_{22}) and cross-group ones (ρ_{12}, ρ_{21}). Even for the simplest system where the N, κ, χ's for the groups are identical, many features are not yet understood. For example, instead of being exponentials, all ρ_{ij}'s are Gaussians. Developing a theory for obtaining their means and widths remains a challenge. Furthermore, we discover surprising phenomena associated with a macroscopic quantity: X, the total number of crosslinks. On short time scales, it is Gaussian distributed about some mean value. But on longer time scales ($O\left(N^2\right)$), this mean value wanders over a considerable part of the allowed range $\left[0, \kappa N^2\right]$! For runs with 10^7 MCS with $N \lesssim 100$, the distribution settles into a relatively flat one with soft cutoffs at both ends. Work is in progress for understanding how the locations and the widths of these cutoffs depend on the control parameters of the system[6].

Finally, let us turn to epidemics on an active network. Simulations with $N = 5000$, $\kappa_{\max} = 25$, $\kappa_{\min} = 10$, $\alpha = 10$ and $n_{SI} = 10$ were performed. Two ways can be used to prevent the system from reaching the absorbing state: When a single infected individual is left in the system, it never recovers, i.e., an 'immortal I'. Alternatively, we could add a very small probability for an S to flip spontaneously to an I, modeling typical diseases which make a resurgence after being 'eradicated.' Here, using the first trick, the 'absorbing state' appears as a non-trivial steady state with an exponential distribution of I's near $O(1)$. Since $N = 5000$, such a steady state will appear to have $f \simeq 0$. For simplicity, we used only 'inflexible' individuals so that, in the healthy state, double exponential $\rho(k)$'s around 25 are found. Running with fixed $\mu = 0.5$ and various λ's, the system typically settles after 10K MCS, and the epidemic can be represented as $f(\lambda)$. In the standard SIS models, $f(\lambda < \lambda_c) = 0$; thereafter, it rises linearly and saturates at $(1+\mu)^{-1}$. Here, its behavior should depend on the fear factor in $\kappa(f)$. The crudest approximation is to set the fraction of the population that recovers per unit time ($\mu f N$) equal to the fraction that is infected. To estimate the latter, we consider the average number of links (κ) an individual has, so that $f\kappa(f)$ is the typical number of individuals who can infect an S. Thus, the rate at which the population is infected is $\left[1 - (1-\lambda)^{f\kappa}\right](1-f)N$. These notions can be summarized in a balance equation

$$f\kappa(f)\ln(1-\lambda) = \ln(1-(1+\mu)f) - \ln(1-f) \tag{2}$$

the solution of which defines $f(\lambda)$. Of course, the details of $\kappa(f)$ are important, but these can be easily tracked. Plotting each side vs. f, we see immediately that $f = 0$ is a solution. However, when λ rises above $\lambda_c \equiv 1 - e^{-\mu/\kappa_{\max}}$, a second crossing appears with $f > 0$. This represents the active epidemic, with $f(\lambda)$ easily obtained numerically. Remarkably, this crude estimate yields good agreement with simulation data for all three fear factors studied. Detailed comparisons will be published elsewhere [7]. More challenging is the task of finding degree distributions in the transition region. Not surprisingly, fluctuations are large and the distributions deviate significantly from double exponentials or Gaussians. Of course, much more work will be needed if we seek a full picture of the epidemic with neither an 'immortal I' nor 'resurgence', so that non-trivial extinction scenarios must be studied.

Concluding Remarks. In this brief article, we report preliminary studies of adaptive networks with preferred degrees and their implications on the SIS model of epidemic spreading. We focus first on networks in which nodes have no associated degrees of freedom, but only act to add/cut links with other nodes. If two groups with different κ's are coupled (such as extroverts and introverts), 'frustration' arises even in the steady state. This aspect has not been explored fully, though the average behavior of how the network reaches the steady state has been reported recently [8]. Here, we are concerned with more details than averages, such as degree distributions. Contrary to naive expectations, these are not simple Gaussians around κ. In the case of a single group (homogeneous network), a simple argument leads us to double exponential tails, in reasonably good agreement with simulations. When a second group is introduced into the population and coupled to the first, a generalization of this argument still provides good agreement, but only for the total degree distributions. Puzzles remain for the distributions of both intra-group and inter-group degrees. For the simple coupling used, the behavior of X, the total number of crosslinks, is not well understood. In particular, even the available phase space for X (e.g., the number of graphs with X links between a group of N_1 nodes and another with N_2 nodes, all with fixed degree κ) is not known analytically. Exploring such 'simple' questions will be helpful for understanding the presence of, say, the very different time scales in our system.

There are obviously many avenues for future research, apart from the unresolved issues posed above. In the case of two networks, the most general way to model the interactions is to introduce two sets of adding/cutting rates, $w_\pm(k_i, k_\times|\kappa, \mu, ...)$, where k_i and $k_\times$ denote the degree of internal and cross links, respectively. Of course, the price paid for such studies is the large dimension of parameter space. In a real society, individual preferences surely fall into a broad spectrum, rather than just two distinct groups with fixed $\kappa_{1,2}$. Do such generalizations mean that degree distributions will become more 'normal' (i.e., Gaussians)? Turning to nodes with their own degrees of freedom, we envisage studying two or more groups (e.g., extroverts and introverts), in order to gain some insight into epidemic spreading/control in active networks. In particular, it would surely be helpful to have a more quantitative understanding of how "Popular People Help Experts Predict Flu Outbreak," headlined in a recent ABC news article [9]. Beyond SIS, many other social networks can be studied (e.g., more complex epidemic models, nodes endowed with wealth or opinions), not to mention the long list of networks listed in the Introduction. Clearly, there will be many more topics for future research in adaptive networks than scientists available to investigate them.

Acknowledgements We thank S. Eubank and T. Platini for illuminating discussions. This research is supported in part by grants from the NSF: DMR-0705152 and DMR-1005417.

References

[1] E. Estrada, M. Fox, D.J. Higham, and G.-L. Oppo (Eds.), *Network Science: Complexity in Nature and Technology* (Springer, New York, 2010).

[2] A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, Cambridge, 2008)

[3] R.M. Anderson and R.M. May, *Infectious Diseases of Humans* (Oxford University Press, New York, 1991).

[4] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Rev. Mod. Phys.* **80** 1275 (2008).

[5] T. Gross, Carlos J. Dommar D'Lima, and B. Blasius, *Phys. Rev. Lett.* **96**, 208701 (2006); L.B. Shaw and I.B. Schwartz, *Phys. Rev. E* **77**, 066101 (2008), and *Phys. Rev. E* **81**, 046120 (2010).

[6] Wenjia Liu, B. Schmittmann and R.K.P. Zia, to be published

[7] Shivakumar Jolad, R.K.P. Zia and B. Schmittmann, to be published

[8] T. Platini and R.K.P. Zia, *J. Stat. Mech.* P10018 (2010).

[9] The full article can be found on http://abcnews.go.com/print?id=11645758.